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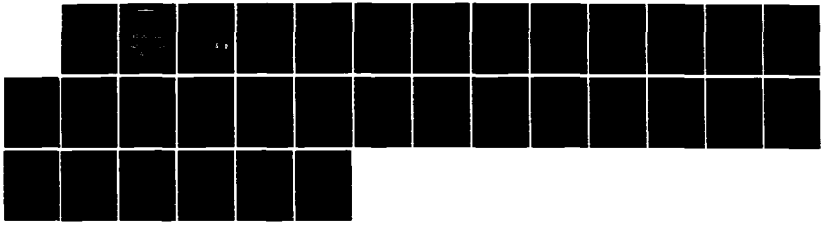

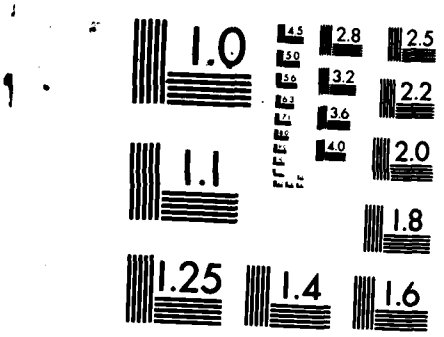

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H. D. Mittelmann

Technical Report No. 88
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Arizona State University
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A PSEUDO-ARCLENGTH CONTINUATION METHOD FOR NONLINEAR EIGENVALUE PROBLEMS

Hans D. Mittelmann†

Abstract. A variant of the classical pseudo-arclength continuation method is proposed. Basically, the method can be viewed as pseudo-arclength continuation in (r,λ)-space where r is a functional of the solution. Another difference is a three-parameter predictor instead of the standard Euler step. This predictor, as well as the Newton corrector iteration, are justified and some numerical results for reaction-diffusion equations are presented. The method provides a simple algebraic check for symmetry-breaking bifurcation, the most common type of secondary bifurcation in physical examples.

Keywords: parameter-dependent boundary value problems, continuation algorithm; singular points; symmetry-breaking bifurcation; reaction-diffusion equations.

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1. Introduction

Boundary value problems for partial differential equations that describe phenomena in many fields of application are usually nonlinear and depend on one or several parameters of the underlying system. These parameters enter in various ways. The coefficients or right-hand sides of the differential equations may depend on them, the boundary conditions or, for example, the domain. With respect to these parameters, in general, bifurcation and non-unicity effects occur. For a list of different problems from the applications as well as a first introduction to analytic and numerical aspects we refer to [8]. A survey of the state of the art may be obtained from [7].

Frequently, the only way to explore the solution manifolds of these problems is to continue along the solution branches, in the case of multi-parameter dependence, with respect to one of the parameters for fixed values of the others. This approach yields a quantitatively satisfactory approximation if, for example, suitable discretizations are used, while it may be difficult to get a full qualitative impression in this way. Other analytic techniques yielding the latter but not also a quantitatively useful result, may well supplement the numerical methods.

For the continuation along solution branches, different methods have been proposed and applied. We restrict considerations here to the methods introducing an arclength-like additional parameter. One of the earlier and frequently referenced papers is [6]. To be more specific, let the

parameter-dependent nonlinear problem be given by

$$G(u,\alpha) = 0 \tag{1.1}$$

where u is an element of a Banach space X and G maps $X \times R^p$ into X. We denote by λ the component of α that is presently not fixed and, for simplicity, suppress the others. The pseudo-arclength method of [6] then introduces an additional parameter σ and augments (1.1) by a normalizing condition

$$\begin{aligned} G(u,\lambda) &= 0, \\ N(u(\sigma),\lambda(\sigma),\sigma) &= 0 . \end{aligned} \tag{1.2}$$

Thus, u as well as λ are locally parametrized by σ. In the following section, we will introduce a slight modification of this approach. In applications, one is frequently interested in a functional of u, for example, a norm, the value of the definite integral of u over some region, etc. This quantity may represent a physically important parameter just as the components of α. For graphical representations also, usually such a functional is depicted versus one or two of the α's. Using a functional of u, therefore, is not equivalent to introducing another artificial parameter.

After introducing the basic continuation method, we discuss a new predictor to be used in conjunction with it in Section 3. An analysis of the corrector iteration is given in the following section. and in the last section, some numerical results are presented.

2. The Continuation Method

For the problem

$$G(u,\lambda) = 0, \quad G: X \times R \to X \tag{2.1}$$

let s denote the arclength along a solution arc $(u(s),\lambda(s))$. Under sufficient smoothness and regularity assumptions this latter dependence is differentiable and thus $(G_u^0 = G_u(u_0,\lambda_0))$

$$G_u^0 \dot{u}_0 + G_\lambda^0 \dot{\lambda}_0 = 0 , \tag{2.2a}$$

$$\|\dot{u}_0\|^2 + \dot{\lambda}_0^2 = 1 \tag{2.2b}$$

in a solution $(u_0,\lambda_0) = (u(s_0),\lambda(s_0))$ of (2.1). Here, subscripts stand for partial derivatives and a dot for differentiation with respect to s.

In [6] it was proposed to augment (2.1) by the following normalizing condition:

$$N_\sigma(u,\lambda,\sigma) \equiv \theta \dot{u}_0(u-u_0) + (2-\theta)\dot{\lambda}_0(\lambda-\lambda_0) - \delta\sigma = 0 \tag{2.3}$$

which approximates (2.2b). σ is thus called the pseudo-arclength parameter. $\delta\sigma$ is the steplength in σ. θ satisfying $0 < \theta < 2$ was meant to be a weighting parameter.

Let r(u) denote a functional of u. For simplicity, let

$$r(u) = \|u\| \tag{2.4}$$

where $\|\cdot\|$ is the norm of u which we assume to be differentiable. Other functionals may be chosen, as for example, in VLSI problems ([2]), but in the following we restrict considerations to (2.4). Quite frequently, this parameter is used in a graphical representation of the solution manifold and also has a physical significance.

The following augmenting equation leads to a pseudo-arclength method in (r,λ)-space.

$$(2.5) \qquad N_r(u,\lambda,\sigma) \equiv \theta\dot{r}_0(r-r_0) + (2-\theta)\dot{\lambda}_0(\lambda-\lambda_0) - \delta\sigma = 0,$$

where $0 \leq \theta \leq 2$, $r_0 = \|u_0\|$, $\dot{r}_0 = \left(\frac{d}{d\sigma}\|u\|\right)_{\sigma=\sigma_0}$.

While the classical pseudo-arclength method with Euler predictor using (2.3) is in general optimal in a local sense among continuation procedures utilizing only first derivatives and depending only on one parameter, in this case the step-length along the tangent, (2.5) will be combined with a more expensive predictor to yield in many examples a better global continuation behaviour. This will be demonstrated in 5.

Let us assume from now on that X is a Hilbert space, sufficient regularity is given, and $\|\cdot\| = (\cdot,\cdot)^{1/2}$, then

$$(2.6) \qquad \dot{r}_0 = (u_0,\dot{u}_0)/r_0 .$$

From (2.5) we see that the three values 0,1,2 for θ are special in the sense that for $\theta = 0$ (2) the augmenting equation characterizes points with a fixed $\lambda(r)$-value, while for $\theta = 1$

the points lie on a hyperplane orthogonal to the solution arc in (r_0, λ_0).

This suggests to continue to target values in the two physical parameters r and λ. Such a strategy seems appropriate as long as starting from a point on the solution curve another point corresponding to a suitably picked target value can be computed in a few number of (corrector) iterations. This will, of course, depend on the starting guess and thus on the method used for predicting this point. Only the combination of (2.5) with the predictor presented in the next section makes this continuation the effective and powerful algorithm it has proven to be at least for a series of problems from the applications.

3. The Predictor

In continuing from a point (u_0, λ_0) on a solution branch most continuation methods first generate a starting guess (u_p, λ_p) and then iteratively determine a new solution point. The pseudo-arclength method in its standard version uses a first order Euler predictor

$$(3.1) \qquad \begin{pmatrix} u_p \\ \lambda_p \end{pmatrix} = \begin{pmatrix} u_0 + \delta\sigma\dot{u}_0 \\ \lambda_0 + \delta\sigma\dot{\lambda}_0 \end{pmatrix} .$$

Subsequently, the augmented system (2.1), (2.3) is iteratively solved ('corrector'). The computed solution for $\theta = 1$ is thus located in the intersection of the solution manifold and a hyperplane perpendicular to the tangent vector $(\dot{u}_0, \dot{\lambda}_0)^T$ at the

previous point and a distance $\delta\sigma$ from this point. It is not immediately apparent how the parameter θ would have to be chosen to improve the quality of the predicted point.

It is clear that with respect to the information used, namely $\dot{u}_0$ and $\dot{\lambda}_0$, the Euler predictor step is locally, i.e., for $\delta\sigma \to 0$, optimal, but that, for example, in highly curved parts of the solution arc, only a rather small $\delta\sigma$ will lead to convergence of the corrector iteration.

In order to improve the quality of the predictor either a higher order method making use of higher derivatives of G may be used or in some other way, it has to be assured that the predicted point better approximates a solution of the nonlinear eigenvalue problem. The first approach is suitable in particular if the nonlinearities have a simple form as, for example, low order polynomials. Here, we propose to make use of only the first order information also used in the Euler predictor. Instead of a single parameter, three are used in

$$\begin{pmatrix} u_p \\ \lambda_p \end{pmatrix} = \begin{pmatrix} (1+\gamma)u_0 + \beta\dot{u}_0 \\ \lambda_0 + \alpha\dot{\lambda}_0 \end{pmatrix} . \tag{3.2}$$

In case $\alpha \neq \beta$, $\gamma = 0$ the predicted point is in a non-tangential direction from (u_0,λ_0). This is slightly more general than (3.1) but, of course, treats λ as a special variable as opposed to methods that, for a finite-dimensional problem, give λ no preferential treatment ([12]). The justification for (3.2) can only be that λ, in fact, represents an important physical parameter.

More explanation is needed for the coefficient of u_0 in (3.2). Before this will be done, we complete the definition of the predictor. A system of three equations is used to determine the parameters α, β and γ.

$$N(u_p, \lambda_p, \sigma) = 0 \ ,$$

$$(u_0, G(u_p, \lambda_p)) = 0 \ , \tag{3.3}$$

$$(\dot{u}_0, G(u_p, \lambda_p)) = 0 \ .$$

In addition to the normalizing equation, two orthogonality relations with respect to the scalar product on X are imposed. The basis for these equations is twofold. First, it assures that the predicted point is a weak solution of (2.1) with respect to the subspace spanned by u_0 and $\dot{u}_0$. On the other hand, (3.2) shows that

$$(u_p, G(u_p, \lambda_p)) = 0 \tag{3.4}$$

holds, which defines a generalized Rayleigh-quotient.

The system (3.3) in general need not have a solution (α, β, γ) or such a solution is not necessarily unique. If, for example, u_0 and $\dot{u}_0$ are parallel, then the rank of the Jacobian of (3.3) is at most two. Therefore, a least squares solution of (3.3) is to be computed. In order to be able to analyze several important features of the continuation method, we have

to more closely consider the case of bifurcation points. Let us introduce the notation

$$F(y) = \begin{pmatrix} G(u,\lambda) \\ N(u,\lambda,\sigma) \end{pmatrix} , \quad y = (u,\lambda) \tag{3.5}$$

and use subscripts for partial derivatives. (u_0,λ_0) is a simple pitchfork (symmetric) bifurcation point of (2.1) if the following conditions hold

$$\begin{aligned} &N(G_u^0) = \text{span}\,\{\phi_0\},\ \|\phi_0\| = 1 , \\ &R(G_u^0) \text{ is closed and codim } R(G_u^0) = 1 , \\ &\phi_0^* G_\lambda^0 = 0 ,\ \phi_0^* G_{uu}^0 \phi_0 \phi_0 = 0 , \\ &b = \phi_0^*(G_{u\lambda}^0 \phi_0 + G_{uu}^0 v_0 \phi_0) \neq 0 \end{aligned} \tag{3.6}$$

where v_0 is the unique solution of

$$G_u^0 v_0 + G_\lambda^0 = 0,\ \phi_0^* v_0 = 0 . \tag{3.7}$$

N, R denote nullspace and range, respectively, and

$$N(G_u^{0*}) = \text{span}\,\{\phi_0^*\} ,\ \phi_0^* \phi_0 = 1 .$$

From differentiating (2.1) twice with respect to s it follows that (cf. [4]) the algebraic bifurcation equation

$$(3.8) \qquad (c\dot{\lambda}_0 + 2b\xi_1)\dot{\lambda}_0 = 0$$

holds where $c = \phi_0^*(G^0_{uu}v_0v_0 + 2G^0_{u\lambda}v_0 + G^0_{\lambda\lambda})$ and $\xi_1^2 + 2\dot{\lambda}_0^2 = 1$. The two solution branches passing through (u_0,λ_0) are thus given by

$$(3.9) \qquad \begin{aligned} u(s) &= u_0 + (s-s_0)(\dot{\lambda}_0 v_0 + \xi_1\phi_0) + O((s-s_0)^2) , \\ \lambda(s) &= \lambda_0 + (s-s_0)\,\dot{\lambda}_0 + O((s-s_0)^2) . \end{aligned}$$

These pitchfork bifurcation points are not generic in the sense that they are in general not present under perturbations of the problem as, for example, introduced by a discretization. An even further specialized class of bifurcation points, the symmetry-breaking bifurcation points, however, have this property.

Let us assume for simplicity that S is a linear operator on X, $S^2 = I$, $S \neq I$ such that

$$(3.10) \qquad G(Su,\lambda) = SG(u,\lambda), \text{ for all } u \in X, \lambda \in R .$$

Then X may be split as

$$(3.11) \qquad X = X_s \oplus X_a ,$$

where $X_s = \{u \in X, u = Su\}$ contains the symmetric elements with respect to S and $X_a = \{u \in X, u = -Su\}$, the anti-symmetric elements. The pitchfork bifurcation point (u_0,λ_0) is then symmetry-breaking if

(3.12) $\quad u_0 \in X_s \ , \quad v_0 \in X_s \ , \quad \phi_0 \in X_a \ .$

On the branch corresponding to the solution $\dot{\lambda}_0 = 0$ of (3.8) it thus holds that

(3.13) $\quad \dot{u}_0 = \pm \phi_0 \in X_a$

while on the other branch $c = 0$ implies $\dot{u}_0 \in X_s$. On the symmetry-breaking branch we have from (2.6) if $S = S^*$

(3.14) $\quad \dot{\lambda}_0 = 0 \ , \quad \dot{r}_0 = 0$

and thus, in this case, N_r in (2.5) is not well defined. The restriction to the two-dimensional (r,λ)-subspace introduces additional singularities that are not present in the classical pseudo-arclength method. A natural modification is to introduce as a parameter the norm of the anti-symmetric part u_a of u corresponding to the splitting (3.11). This was done explicitly in [10]. Since on the branch of nonsymmetric solutions $u_a = \dot{u}_0 \in X_a$, asymptotically in the bifurcation point from (3.9), we propose instead in the general case to use $N = N_\sigma$ in (3.3) whenever $|\dot{r}_0|/\|\dot{u}_0\|$ is small.

Let us assume now that starting at (u_0,λ_0) a point (u,λ) on the symmetry-breaking branch with $r = \|u\| < r_0 = \|u_0\|$ is to be computed. For $\gamma = 0$ (3.2), (3.12), (3.13) imply that always $\|u_p\| > r_0$. This is one of the reasons that γ was

introduced in (3.2). Another reason is that in the linear eigenvalue problem, computations along the eigenfunction branches require a renormalization only and are not prone to cancellation if in the predictor step u_p is actually obtained by renormalization, i.e., $\beta = 0$ in (3.2).

Let us summarize the normalizing condition that is proposed in the various cases.

$$
\begin{aligned}
&|\dot{r}_0| < \varepsilon \|\dot{u}_0\|: && N = N_\sigma,\ \theta \neq 0 \\
&|\dot{r}_0| > (1-\varepsilon)\|\dot{u}_0\|: && N = N_r, \theta \neq 0,\ \beta = 0\ , \\
&\varepsilon \|\dot{u}_0\| \leq \dot{r}_0 \leq (1-\varepsilon)\|\dot{u}_0\|: && N = N_r,\ 0 \leq \theta \leq 2\ .
\end{aligned}
\tag{3.15}
$$

In the following section, we will address the question of the regularity of the corrector Jacobian in the various cases of (3.15).

4. The Corrector

The point (u_p, λ_p) computed by the predictor described in the previous section is used as usual as starting guess for an iterative solution of the augmented system (3.5). This is done by Newton's method and thus the Jacobian

$$
F_y = \begin{pmatrix} G_u & G_\lambda \\ N_u & N_\lambda \end{pmatrix} \tag{4.1}
$$

has to be evaluated and inverted. The following lemma is a simplified version of Lemma 2.8 in [6].

Lemma 4.1 Let X be a Banach space and consider the linear operator $\bar{A}$: XxR → XxR of the form

$$\bar{A} = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}$$

where A: X → X, B: R → X, C*: X → R, D: R → R .

(i) If A is nonsingular then $\bar{A}$ is nonsingular iff

(4.2) $E = D - C^*A^{-1}B$ is nonzero.

(ii) If A is singular and dim N(A) = codim R(A) = 1 then $\bar{A}$ is nonsingular iff

(4.3) $B \notin R(A)$, $C^* \notin R(A^*)$.

Proposition 4.2 For an appropriate choice of the continuation parameter, i.e. λ respectively r, F_y in (4.1) is regular except at bifurcation points.

Proof The quantity E in (4.2) is easily computed as $E = (2-\Theta)\dot{\lambda}_0 + \Theta\dot{r}_0^2/\dot{\lambda}_0$ if $N = N_r$. If we accept as an appropriate choice of the continuation parameter not to pick r respectively λ to continue past a turning point in r respectively λ then we have $\Theta \neq 0$ if $\dot{\lambda}_0 = 0$ respectively $\Theta \neq 2$ if $\dot{r}_0 = 0$, so

that F_y is regular in regular points for $N = N_r$. In bifurcation points $G_\lambda \in R(G_u)$ so that F_y is always singular from (4.3) for $N = N_r$ and $N = N_\sigma$. It remains to consider turning points.

If $\dot\lambda_0 = 0$ and $G_\lambda \notin R(G_u)$ then the last condition in (4.3) is equivalent to $(\theta\dot r_0 u_0/r_0, \dot u_0) = \theta\dot r_0^2 \neq 0$. So if $\theta \neq 0$, $r_0 \neq 0$ then F_y is regular for $N = N_r$ unless $\dot r_0 = 0$. This situation occurs in symmetry-breaking pitchfork bifurcation points. These points appear to be the most common type of secondary bifurcation in physical examples. Finite dimensional problems exhibiting secondary bifurcation often arise from symmetry preserving discretizations of such continuous problems. It is, therefore, important that the continuation method can handle these points. In addition to that, the proposed algorithm allows to detect these points by checking (3.14). We refer to Example 1 in [11] where two symmetry-breaking bifurcation points are computed and to the second example in 5. According to (3.15) $N = N_\sigma$ is chosen in these points since N_r is not well defined. If the point is not a bifurcation point both conditions of (4.3) are satisfied and F_y is again regular.

The above proposition does not specify in any more detail what happens in bifurcation points. If we consider F_y^{-1} in the neighbourhood of a simple bifurcation point $y = y(s_0)$ and measure the growth of $\|F_y^{-1}\|$ as a function of $(s-s_0)^{-1}$ as in [4] then we obtain the following result along the lines of proofs in [4,5].

Proposition 4.3 If the normalization condition N is chosen according to (3.15) then on any of the branches passing through a simple bifurcation point $y_0 = (u(s_0), \lambda(s_0))$ of (2.1) for $\delta > 0$ sufficiently small

$$(4.4) \qquad \|F_y^{-1}(y(s))\| = O(|s-s_0|^{-1}), \quad 0 < |s-s_0| < \delta .$$

The growth behaviour of (4.4) allows the application of the Newton-Kantorovich theorem to show convergence for the starting point obtained by the Euler predictor ([4]). For the combination with the predictor (3.2), (3.3) a different analysis would have to be done. One of the facts that complicate such an analysis considerably is the least squares solution of (3.3). We do not present such a theory here and refer to an illustration of the practical merits of the method by the numerical results in [11] and in the following section.

5. Numerical Results

We present here only a demonstration of the behaviour of the proposed continuation method. The method was developed and implemented in the program PLTMG [1] in cooperation with R. Bank. The class of second order nonlinear parameter-dependent boundary value problems solved by PLTMG includes many interesting applications.

It is of interest how the continuation method proposed above compares to standard methods as, for example, the

classical pseudo-arclength method, $\theta = 1$ in (2.3). The generalized inverse iteration ([9]) that strongly influenced the method presented here was compared to different continuation methods in [9] and to a predecessor of the present method using a two-parameter predictor and (2.5) in [10]. The latter paper also contains comparisons with a variant of the generalized inverse iteration in the neighbourhood of symmetry-breaking bifurcation points.

An extensive comparison of the present method with the pseudo-arclength method is beyond the scope of this paper. We restrict the consideration to one example exhibiting turning points and thereby illustrate the advantages of a multi-parameter predictor. The second example has a more complicated solution structure and serves to demonstrate the computation of singular points, the switching of branches at bifurcation points, the detection of symmetry-breaking bifurcation points and the switching of continuation parameters in a multi-parameter problem.

The first example is the two-parameter Bratu problem

$$\Delta u + \lambda \exp(u/(1+\varepsilon u)) \text{ in } \Omega \tag{5.1}$$

on the unit square Ω with homogeneous Dirichlet boundary conditions. This problem was solved frequently (see, for example, [3,11]). For $\varepsilon = .1$ the solution curve has two turning points and we present results for continuing up to and beyond the

first one. We point out that this turning point is far from being nearly angular and therefore does not cause particular difficulties for the pseudo-arclength method. For a case of a nearly angular turning point, see Example 2 in [11].

On a standard triangulation of Ω, with 25 vertices, we have discretized (5.1) by linear finite elements. The resulting finite-dimensional system was first solved by the pseudo-arclength method starting at the origin. We assumed that the step-picking procedure does not cut back the steplength α in the Euler predictor, i.e., $\alpha = \beta, \gamma = 0$ in (3.2), enough near the turning point to avoid failure of the corrector iteration. In this latter case, the steplength is cut back by .5 until the corrector converges. In the last column of Table 5.1 the number of corrector iterations is listed and an asterisk denotes failure to converge after the given number of iterations while two asterisks denote failure to yield a descent direction for the damped Newton process.

Subsequently, we have used our continuation method to continue from each of the points in Table 5.1 below the turning point directly to the point above that with "u" = 2.61 which was finally reached. The results in Table 5.2 show that at most 3 corrector steps are needed. The three predictor parameters are also given to show for this example the effect of introducing γ and the early 'detection' of the turning point

through a negative α. We note that for the coarse discretization used the first turning point in case $\varepsilon = .1$ is at $(\lambda, r) = (.840, .928)$.

λ	r	$\dot{\lambda}$	$\dot{r}$	α	it
0	0	1	0	1	1
1	.04	.999	.042	5	2
6	.343	.995	.095	2	3
8	.646	.959	.282	.665	10*
				.332	2
8.28	.772	.847	.531	.920	10*
				.460	10**
				.230	10**
				.115	2
8.37	.848	.596	.803	1.49	10*
				.745	10*
				.373	10*
				.186	4**
				.093	2
8.40	.943	-.125	.991	2.09	4
4.91	2.61	-.894	.447		

Table 5.1 Continuation with the pseudo-arclength method for (5.1), $\varepsilon = .1$

λ	$\dot{\lambda}$	α	β	γ	it
1	.999	4.40	.17E-3	64.3	3
6	.995	- .732	-.21E-2	6.58	3
8	.959	-2.96	-.51E-1	3.01	2
8.28	.847	-3.70	-.125	2.47	2
8.37	.596	-5.54	-.345	2.40	2

Table 5.2 Continuation with proposed method for each point below turning point in Table 5.1 directly to solution with norm r = 2.61

A suitable step-picking procedure may be able to reduce the number of failing corrector iterations for the pseudo-arclength method. But the fact remains that several small steps are necessary to overcome the turning point. The next example has a more complex solution diagram including bifurcation. It illustrates several other features of the continuation method and its implementation.

Let Ω_ε be a domain that is obtained by connecting two regions in the left and right half-plane situated symmetrically w.r.t. the y-axis by a channel of width ε such that Ω_ε is symmetric, too. Homogeneous Neumann boundary conditions are prescribed along $\partial\Omega_\varepsilon$.

$$(5.1) \qquad \frac{\partial u}{\partial t} = \Delta u + \lambda u - a u^2 - u^3 \quad \text{in} \quad \Omega_\varepsilon \ ,$$

The applications of this reaction-diffusion equation include in particular a selection-migration model from population genetics (cf.[13]). We are interested in the stationary solutions only. The bifurcation behaviour of (5.1) w.r.t. the three parameters ε, λ, a is given in [13]. For $a = 0$, for example, secondary bifurcation takes place from the branch bifurcating from the trivial solution at the first nonzero eigenvalue λ_ε of the linearization of (5.1). For $a \neq 0$, and for this case some results will be presented below, secondary symmetry-breaking bifurcation occurs from the branch of constant solutions bifurcating at the origin.

Because of symmetry and the homogeneous Neumann boundary condition, it suffices to take as Ω_ε a non-convex, say, L-shaped domain with vertices (0,0), (1,0), (1,.5), (.5,.5), (.5,1), (0,1). Thus, ε is fixed and the parameters a and λ are left. For $a > 0$ the following schematic bifurcation diagrams were obtained in [13] for a point value of u versus λ.

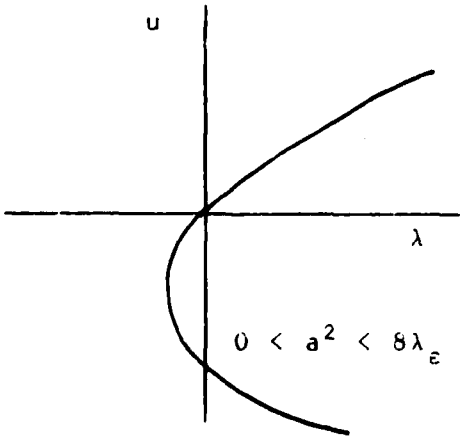

Figure 5.1a

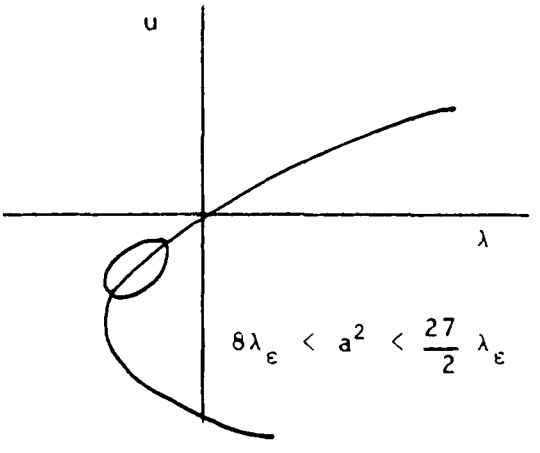

Figure 5.1b

The elliptic version of (5.1) was discretized on a relatively coarse triangulation Ω_h with 21 vertices and linear finite elements were used to compute the approximate solution u_h. Better approximations at specified points of the solution branches may be obtained through PLTMG by a multi-grid method.

The first nontrivial eigenvalue was $\lambda_{\varepsilon h} = 6.73$. In fact, for $a = 5$ no secondary bifurcation was observed. For $a = 9$ the two secondary bifurcations points in Figure 5.1b are located at $(\lambda_1, r_1) = (-8.17, 1.04)$ and $(\lambda_2, r_2) = (-14.6, 2.42)$, while the turning point is at $(\lambda_3, r_3) = (-16, 3.46)$.

We present below the output of PLTMG for the following continuation. Starting on the trivial solution in $\lambda = 0$ we switch branches three times using Method I of [6]. On the branch of constant negative solutions we continue to (λ_1, r_1). There we switch branches once and continue on this branch of non-constant solutions to $(\lambda, r) = (-11.5, 1.6)$. Now we keep this λ fixed and switch to a as continuation parameter. From the value $a = 8$ we continue to smaller values. We expect to first come to the constant solutions and continuing further to reach a point where no further continuation is possible. The first happens between $a = 7.5$ and $a = 7.4$, as a more detailed computation shows, while the branch ends at $a = 6.78$. If instead we continue in r, then $(a, r) = (6.78, 2.94)$ is a turning point w.r.t. a as expected.

it	λ/a	$\|u\|$	$\dot{\lambda}$	$\dot{r}$	$\det(G_u)$	δ
Switch branches three times.						
1	0.000E+00	0.000E+00	0.100E+01	0.000E+00	-0.105E+00	-0.124E-05
0	0.000E+00	0.000E+00	0.994E+00	0.108E+00	-0.105E+00	-0.124E-05
0	0.000E+00	0.000E+00	-0.100E-01	0.000E+00	-0.105E+00	-0.124E-05
0	0.000E+00	0.000E+00	-0.994E+00	-0.108E+00	-0.105E+00	-0.124E-05
Continue to r = 1, then to r = 1.5 and check sign of determinant.						
1	-0.790E+01	0.100E+01	-0.989E+00	0.150E+00	-0.902E+02	0.435E-02
1	-0.109E+02	0.150E+01	-0.982E+00	0.188E+00	0.541E+03	-0.316E-01
Determinant changes sign; find bifurcation point by secant method on δ.						
1	-0.860E+01	0.111E+01	-0.988E+00	0.157E+00	-0.128E+03	-0.651E-02
1	-0.794E+01	0.101E+01	-0.989E+00	0.151E+00	-0.766E+02	0.370E-02
1	-0.818E+01	0.104E+01	-0.988E+00	0.153E+00	0.354E+01	-0.174E-03
Switch branches and continue to r = 1.2, r = 1.6.						
0	-0.817E+01	0.104E+01	0.484E-01	-0.639E-02	0.815E-01	-0.427E-05
2	-0.908E+01	0.116E+01	-0.980E+00	0.142E+00	-0.652E+03	0.255E-01
1	-0.931E+01	0.120E+01	-0.982E+00	0.148E+00	-0.834E+03	0.312E-01
1	-0.115E+02	0.160E+01	-0.976E+00	0.220E+00	-0.197E+04	0.633E-01
Switch continuation parameter to a. Continue to a = 7.5, then to r = 3 and check sign of determinant.						
1	0.800E+01	0.160E+01	0.829E+00	-0.419E+00	-0.197E+04	0.633E-01
3	0.750E+01	0.187E+01	0.393E+00	-0.200E+00	-0.183E+03	0.876E-02
1	0.678E+01	0.300E+01	-0.481E-01	-0.999E+00	0.162E+04	0.178E-01
Determinant changes sign; find turning point by secant method on $\det(G_u)$.						
1	0.679E+01	0.277E+01	0.146E+00	-0.989E+00	-0.218E+04	-0.469E-01
1	0.678E+01	0.294E+01	-0.398E-02	-0.100E+01	0.110E+03	0.142E-02
1	0.678E+01	0.294E+01	-0.307E-03	-0.100E+01	0.836E+01	0.109E-03
1	0.678E+01	0.294E+01	0.844E-05	-0.100E+01	-0.232E+00	-0.300E-05

Table 5.1 Continuation results for problem (5.1)

In Table 5.1, it denotes the number of (damped) Newton corrector iterations, while δ is a quantity that changes sign at simple bifurcation points. We point out that after switching onto the branch of non-constant solutions, $\dot{\lambda}$ and $\dot{r}$ are both small, as expected, indicating a symmetry-breaking bifurcation point (cf.(3.14)).

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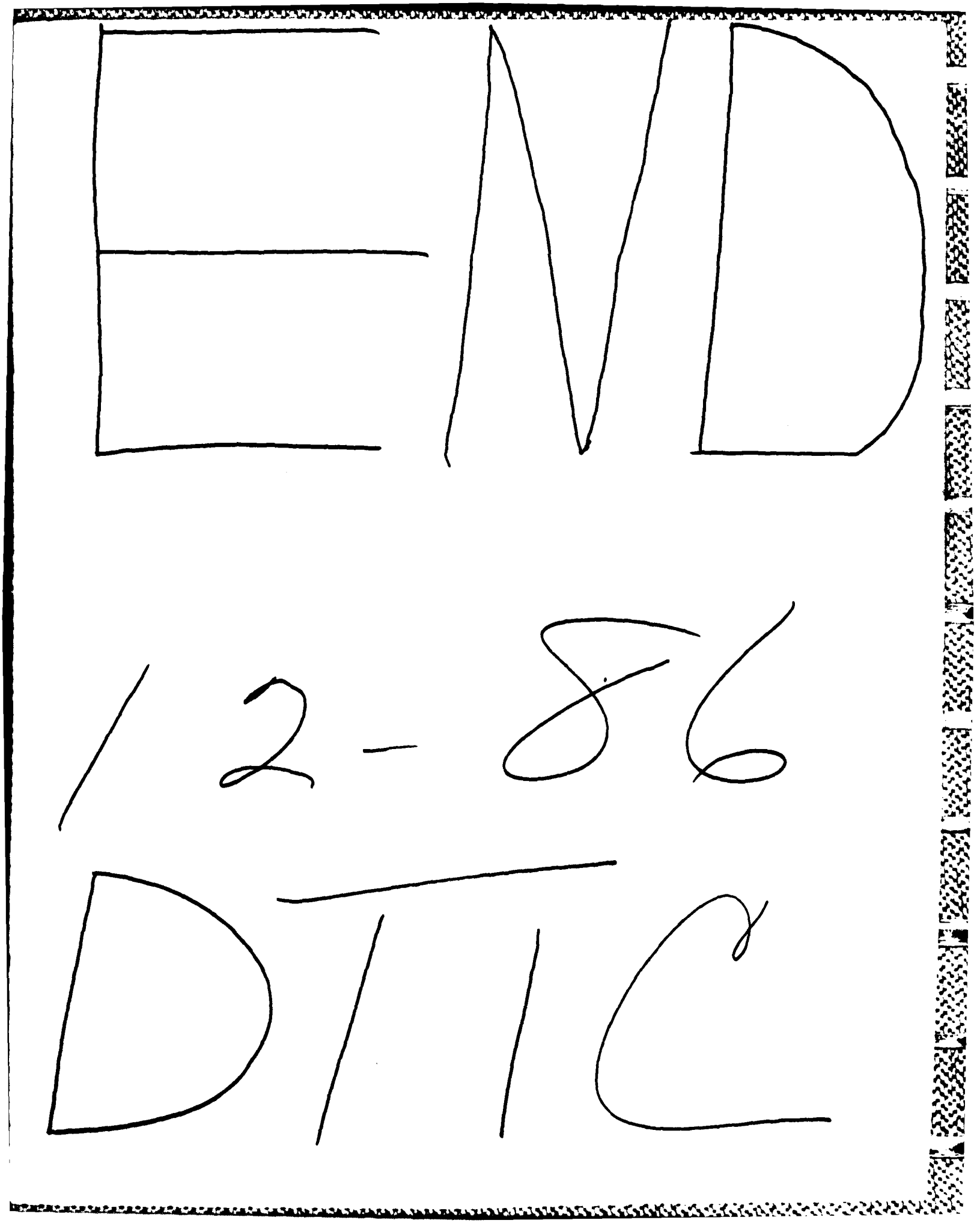
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